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Dynamics of money

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We present a dynamical many-body theory of money in which the value of money is a time dependent ''strategic variable'' that is chosen by the individual agents. The value of money in equilibrium is not fixed by the equations, and thus represents a continuous symmetry. The dynamics breaks this continuous symmetry by fixating the value of money at a level which depends on initial conditions. The fluctuations around the equilibrium, for instance in the presence of noise, are governed by the ''Goldstone modes'' associated with the broken symmetry. The idea is illustrated by a simple network model of monopolistic vendors and buyers. [S1063-651X(99)12308-0]

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I. INTRODUCTION

In classical equilibrium theory in economics [1], agents submit their demand-versus-price functions to a ''central agent'' who then determines the relative prices of goods and their allocation to individual agents. The absolute prices are not fixed, so the process does not determine the value of money, which merely enters as a fictitious quantity that facilitates the calculation of equilibrium. Thus, traditional equilibrium theory does not offer a fundamental explanation of money, perhaps the most essential quantity in a modern economy.

Indeed, a ''search-theoretic'' approach to monetary economics has been proposed [2–4]. Agents may be either money traders, producers, or commodity traders. They randomly interact with each other, and they decide whether or not to trade based on ''rational expectations'' about the value of a transaction. After a transaction the agent changes into one of the two other types of agents. This theory has a steady state where money circulates. As other equilibrium theories, this theory does not describe a dynamics leading to the steady state, of sufficient detail for one to simulate it.

In equilibrium theory, all agents act simultaneously and globally. In reality, agents usually make decisions locally and sequentially. Suppose an agent has apples and wants oranges. He might have to sell his apples to another agent before he buys oranges from a third agent: hence money is needed for the transaction, supplying liquidity. It stores value between transactions.

Money is essentially a dynamical phenomenon, since it is intimately related to the temporal sequence of events. Our goal is to describe the dynamics of money utilizing ideas and concepts from theoretical physics and economics, and to show how the dynamics may fix the value of money.

We study a network of vendors and buyers, each of whom has a simple optimization strategy. Whenever a transaction is considered, the agent must decide the value of the goods and services in question, or, equivalently, the value of money relatively to that of the goods and services he intends to buy or sell. He will associate that value to his money that he believes will maximize his utility. Thus, the value of money is a ''strategic variable'' that the agent in principle is free to choose as he pleases. However, if he makes a poor choice he will loose utility.

For simplicity, we assume that agents are rather myopic: they have short memories, and they take into account only the properties of their ''neighbors,'' i.e., the agents with which they interact directly. They have no idea about what happens elsewhere in the economy.

Despite the bounded rationality of these agents, the economy self-organizes to an equilibrium state where there is a spatially homogeneous flow of money. Since we define the dynamics explicitly, we are, however, also able to treat the nature of this relaxation to the equilibrium state, as well as the response of the system to perturbations and to noise-induced fluctuations around the equilibrium. These phenomena are intimately related to the dynamics of the system, and cannot be discussed within any theory concerned only with the equilibrium situation.

Our model is a simple extension of Jevons' [5] example of a three agent, three commodity economy with the failure of the double coincidence of wants, i.e., when only one member of a trading pair wants a good owned by the other. A way out of the paradox of no trade where there is gain to be obtained by all, is to utilize a money desired by and held by all. Originally this was gold, but here we show that the system dynamics can attach value to ''worthless'' paper money.

We find that the value of money is fixed by a ''bootstrap'' process: agents are forced to accept a specific value of money, despite this value's global indeterminacy. The value of money is defined by local constraints in the network, not by trust. By ''local,'' we simply mean that each agent interact only with a very small fraction of other agents in his neighborhood.

This situation is very similar to problems with continuous symmetry in physics. Consider, for instance, a lattice of interacting atoms forming a crystal. The crystal's physical properties, including its energy, are not affected by a uniform translation X of all atoms, this translational symmetry is continuous. Nevertheless, the position $x(n)$ of the nth atom is restricted by the position of its neighbors. This broken continuous symmetry results in slow, large-wavelength fluctuations, called *Goldstone modes* [6,7] or ''soft modes.'' These modes are easily excited thermally, or by noise, and thus gives rise to large positional fluctuations.

II. MODEL

In our model, we consider N agents, $n=1,2,\ldots,N$, placed on a one-dimensional lattice with periodic boundary conditions. This geometry is chosen in order to have a simple and specific way of defining who is interacting with whom. The geometry is not important for our general conclusions concerning the principles behind the fixation of prices.

We assume that agents cannot consume their own output, so in order to consume they have to trade, and in order to trade they need to produce. Each agent produces a quantity q_n, of one good, which is sold at a unit price p_n, to his left neighbor $n-1$. He next buys and consumes one good from his neighbor to the right, who subsequently buys the good of *his* right neighbor, etc., until all agents have made two transactions. This process is repeated indefinitely, say, once per day.

For simplicity, all agents are given utility functions of the same form

$$u_n=-c(q_n)+d(q_{n+1})+I_n(p_nq_n-p_{n+1}q_{n+1}). \tag{1}$$

The first term, $-c$, represents the agent's cost, or displeasure, associated with producing q_n units of the good he produces. The displeasure is an increasing function of q, and c is convex, say, because the agent gets tired. The second term d, is his utility of the good he can obtain from his neighbor. Its marginal utility is decreasing with q, so d is concave. This choice of c and d is common in economics; see, e.g., [3].

An explicit example is chosen for illustration and analysis,

$$c(q_n)=aq_n^{\alpha}, \quad d(q_{n+1})=bq_{n+1}^{\beta}. \tag{2}$$

The specific values of a, b, α, and β are not important for the general results, as long as c remains convex and d concave. For our analysis we choose $a=\frac{1}{2}$, $b=2$, $\alpha=2$, and $\beta=\frac{1}{2}$.

The last term represents the change in utility associated with the gain or loss of money after the two trades. Notice that the dimension of I_n is [utility per unit of currency], i.e., the physical interpretation is the *value of money*.

Each agent has knowledge only about the utility functions of his two neighbors, as they appeared the day before. The agents are monopolistic, i.e., agent n sets the price of his good, and agent $n-1$ then decide how much q_n, he will buy at that price. This amount is then produced and sold—there is no excess production. The goal of each agent is to maximize his utility, by adjusting p_n and q_{n+1}, while maintaining a constant (small) amount of money. Money has value only as liquidity. There is no point in keeping money, all that is needed is what it takes to complete the transactions of the day.

Thus, the agents aim to achieve a situation where the expenditures are balanced by the income:

$$p_nq_n-p_{n+1}q_{n+1}=0. \tag{3}$$

When the value of money is fixed, $I_n=I$, the agents optimize their utility by charging a price

$$p=2^{1/3}I^{-1} \tag{4}$$

and selling an amount

$$q=2^{-2/3} \tag{5}$$

at that price. This is the monopolistic equilibrium.

Note that the resulting quantities q, are independent of the value of money, which thus represents a continuous symmetry. There is nothing in the equations that fixes the value of money and the prices. Mathematically, the continuous symmetry expresses the fact that the equations for the quantities are ''homogeneous of order one.'' The number of equations is one less than the number of unknowns, leaving the value of money undetermined. We shall see how this continuous symmetry eventually is broken by the dynamics.

Agent n tries to achieve his goal by estimating the amount of goods q_n, that his neighbor will order at a given price, and the price p_{n+1}, that his other neighbor will charge at the subsequent transaction.

Knowing that his neighbors are rational beings like himself, he is able to deduce the functional relationship between the price p_n, that he demands and the amount of goods q_n, that will be ordered in response to this. Furthermore, he is able to estimate the size of p_{n+1}, based on the previous transaction with his right neighbor. This enables him to decide what the perceived value of money should be, and hence how much he should buy and what his price should be. This process is then continued indefinitely, at times $\tau=1,2,3,\ldots$.

This defines the game. The strategy we investigate contains the assumption that agents do not change their valuation of money I, between their two daily transactions, and they maximize their utility accordingly.

The process is initiated by choosing some initial values for the I's. They could, e.g., be related to some former gold standard.

In fixing his price at his first transaction of day τ, agent n exploits the knowledge he has of his neighbors' utility functions, i.e., he knows that the agent to the left will maximize his function with respect to $q_{n,\tau}$

$$\frac{\partial u_{n-1,\tau}}{\partial q_{n,\tau}}=0; \tag{6}$$

hence the left neighbor will order the amount

$$q_{n,\tau}=(I_{n-1,\tau}p_{n,\tau})^{-2}. \tag{7}$$

This functional relationship between the amount of goods $q_{n,\tau}$, ordered by agent $n-1$ at time τ and the price $p_{n,\tau}$, set by agent n, allows agent n to gauge the effect of his price policy. Lacking knowledge about the value of $I_{n-1,\tau}$, agent n instead estimates it to equal the value it had in the previous transaction $I_{n-1,\tau-1}$, which he knows. Eliminating $q_{n,\tau}$ from Eq. (1) we obtain

$$u_{n,\tau}=-\frac{1}{2}I_{n-1,\tau-1}^{-4}p_{n,\tau}^{-4}+2\sqrt{q_{n+1,\tau}}$$
$$+I_{n,\tau}(p_{n,\tau}^{-1}I_{n-1,\tau-1}^{-2}-p_{n+1,\tau}q_{n+1,\tau}). \tag{8}$$

Maximizing this utility $u_{n,\tau}$, with respect to $p_{n,\tau}$ and $q_{n+1,\tau}$ yields

$$p_{n,\tau}=2^{1/3}I_{n,\tau}^{-1/3}I_{n-1,\tau-1}^{-2/3}, \tag{9}$$

and

$$q_{n+1,\tau}=(I_{n,\tau}p_{n+1,\tau})^{-2}. \tag{10}$$

By arguments of symmetry,

$$p_{n+1,\tau}=2^{1/3}I_{n+1,\tau}^{-1/3}I_{n,\tau-1}^{-2/3} \tag{11}$$

is the price agent $n+1$ will demand of agent n in the second transaction. Since agent n does not yet know the value of $I_{n+1,\tau}$, he instead uses the known value of $I_{n+1,\tau-1}$ when estimating $p_{n+1,\tau}$.

In the constraint Eq. (3), the following expressions are used:

$$q_n=q_{n,\tau}^{(\text{guess})}=(I_{n-1,\tau-1}p_{n,\tau})^{-2}, \tag{12}$$

$$p_{n+1}=p_{n+1,\tau}^{(\text{guess})}=2^{1/3}I_{n+1,\tau-1}^{-1/3}I_{n,\tau-1}^{-2/3}, \tag{13}$$

$$q_{n+1}=q_{n+1,\tau}^{(\text{guess})}=(I_{n,\tau}p_{n+1,\tau}^{(\text{guess})})^{-2}, \tag{14}$$

and p_n is given by Eq. (9). Solving for $I_{n,\tau}$, and evaluating at time $\tau+1$, we find [8]

$$I_{n,\tau+1}=(I_{n-1,\tau}^{4}I_{n,\tau}^{2}I_{n+1,\tau})^{1/7}, \tag{15}$$

which sets agent n's value of money on day $\tau+1$ equal to a weighted geometric average of the value agent n and his two neighbors prescribed to their money the previous day. Using this value of I_n, agent n can fix his price p_n and decide which quantity q_{n+1}, he should optimally buy. This simple equation completely specifies the dynamics of our model. The entire strategy can be reduced to an update scheme involving only the value of money—everything else follows from this. Thus, the value of money can be considered the basic strategic variable.

Although Eq. (15), has been derived for a specific simple example, we submit that the structure is much more general. In order to optimize his utility function, the agent is forced to accept a value of money, and hence prices, which pertain to his economic neighborhood. Referring again to a situation from physics, the position of an atom on a general lattice is restricted by the positions of its neighbors, despite the fact that the entire lattice can be shifted with no physical consequences.

Even though there is no utility in the possession of money, as explicitly expressed by Eq. (3), the strategies and dynamics of the model nevertheless leads to a value being ascribed to the money. The dynamics in this model is driven by the need of the agents to make estimates about the coming transactions. In a sense, this models the real world where agents are forced to make plans about the future, based on knowledge about the past—and, in practise, only a very limited part of the past. In short: the dynamics is generated by the bounded rationality of the agents.

In the steady state, where the homogeneity of the utility functions give $I_n=I_{n+1}$, we retrieve the monopolistic equilibrium equations (4) and (5).

III. SOLVING THE DYNAMICS

Taking the logarithm in Eq. (15) and introducing $h_{n,\tau}=\ln(I_{n,\tau})$ yields the linear equation:

$$h_{n,\tau+1}=\frac{4}{7}h_{n-1,\tau}+\frac{2}{7}h_{n,\tau}+\frac{1}{7}h_{n+1,\tau}, \tag{16}$$

describing a Markov process. Now assume that $h_{n,\tau}$ is a slowly varying function of (n,τ) and that we may think of it as the value of a differentiable function $h(x,t)$ in $(x,t)=(n\,\delta x,\tau\,\delta t)$. Then, expanding to first order in δt and second order in δx, we find the diffusion equation

$$\frac{\partial h(x,t)}{\partial t}=D\frac{\partial^2 h(x,t)}{\partial x^2}-v\frac{\partial h(x,t)}{\partial x}, \tag{17}$$

with diffusion coefficient $D=\frac{5}{14}(\delta x)^2/\delta t$, and convection velocity $v=\frac{3}{7}\delta x/\delta t$. The generator T, of infinitesimal time translations is defined by

$$\frac{\partial h(x,t)}{\partial t}=Th(x,t). \tag{18}$$

Taking the lattice Fourier transformation, the eigenvalues of T are found to be $\lambda_k=-k^2D-ikv$, where the periodic boundary condition yields $k=(2\pi/N)l$; $l=0,1,\ldots,N-1$. The damping time for each mode k, is given by $t_k=(k^2D)^{-1}$, i.e., it increases as the square of the system size N. The only mode that is not dampened has $k=0$, and is the soft ''Goldstone mode'' [6,7] associated with the broken continuous symmetry with respect to a uniform shift of the logarithm of prices in the equilibrium:

All prices can be changed by a common factor, but the amount of goods traded will remain the same. The rest of the modes are all dampened (for a finite-size system), and hence the system eventually relaxes to the steady state.

Figures 1 and 2 show results from a numerical solution of Eq. (16) for 1000 agents with random initial values for the variables h (sampled from a uniform distribution on the interval [0,2]). Figure 1 shows the spatial variation of prices at two different times—convection with velocity $v=\frac{3}{7}\delta x/\delta t$ is clearly seen, while the effect of diffusion is not visible on this time scale. The relatively weak effect of diffusion means that spatial price variations, such as those shown in Fig. 1, can travel around the entire lattice many times before diffusion has evened them out. Consequently, the individual agent experiences price oscillations with slowly decreasing amplitude, as seen in Fig. 2.

Thus, despite the myopic behavior of agents, the system evolves towards an equilibrium. But in contrast to equilibrium theory, we obtain the temporal relaxation rates towards the equilibrium, as well as specific absolute values for individual prices. The value of money is fixed by the history of the dynamical process, i.e., by the initial condition combined with the actual strategies by the bounded rational agents.

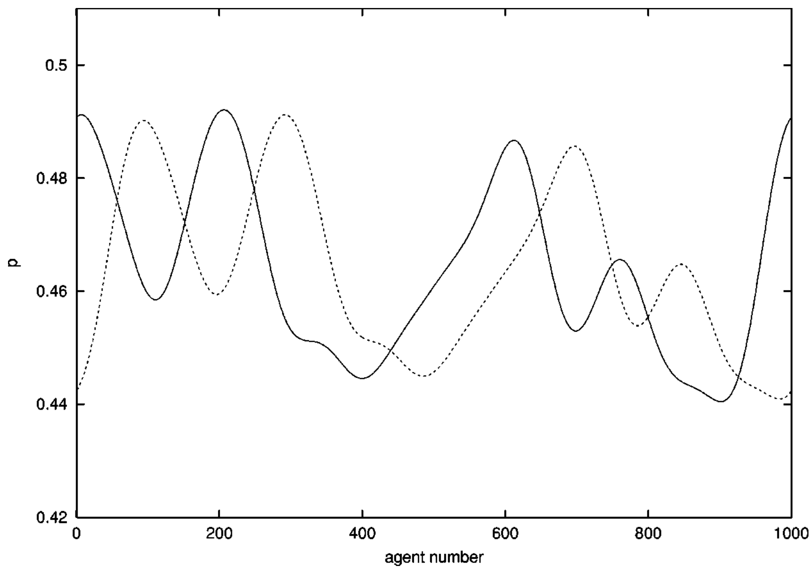

FIG. 1. Variation of prices for all agents at two different times, $\tau = 3000$ (solid line) and $\tau = 3200$ (dotted line).

IV. NOISE

If an agent is suddenly supplied with some extra amount of money, he will lower his value of money, hence increase his price and consequently work less and buy more goods, the effect being inflation propagating through the system, as described by the solution to Eq. (17) for a delta-function initial condition [9]. Likewise, the destruction or loss of some amount of money by a single agent will affect the whole system. These are both transient effects, and in the steady state the same amount of goods will be produced and consumed, as before the change.

In general, there might be some noise in the system, due to imperfections in the agents' abilities to optimize properly their utility functions, or due to external sources affecting the utility functions. A random multiplicative error in estimating the value of money transforms to a linear noise in Eq. (17). We assume that the noise $\eta(x,t)$, has the characteristics: $\langle \eta(x,t) \rangle = 0$ and $\langle \eta(x,t)\, \eta(x',t') \rangle = A\,\delta(x-x')\,\delta(t-t')$. Adding it to Eq. (17) and taking the Fourier transform (with periodic boundary conditions in a system of size L) one finds the equal-time correlation function:

$$\langle [h(x)-h(0)][h(x')-h(0)] \rangle = \frac{A}{2DL} \sum_q q^{-2}(e^{iqx}-1)(e^{-iqx'}-1), \qquad (19)$$

where $q=(2\pi/L)\,n$; $n=0,\pm 1,\pm 2,\ldots$. For $x=x'$ and $L \to \infty$ this becomes

$$\langle [h(x)-h(0)]^2 \rangle = \frac{A}{2D} x, \qquad (20)$$

viz., the dispersion for a biased random walker in one dimension with position h, time x, and diffusion coefficient $A/4D$. In the presence of noise, the agents no longer agree about the value of money, and there will be large price fluctuations. The fluctuations reflect the lack of global restoring force due to the continuous global symmetry.

How much money is needed to run an economy? In this model-economy the total amount of money is reflected in the agents' I's, and is always conserved, as seen by

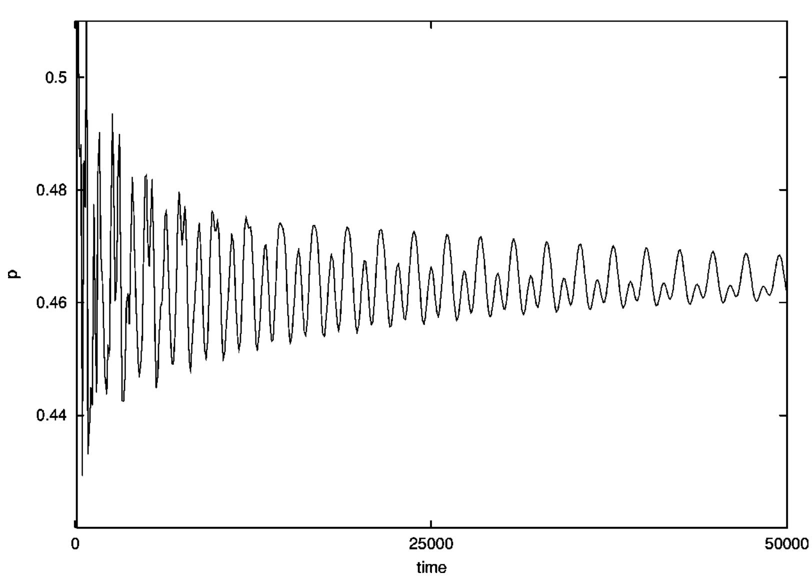

FIG. 2. Price variation for a single agent. The oscillations are an artifact due to the periodic boundary condition, setting $h_{N+1,\tau} = h_{1,\tau}$.

$$\sum_n (p_n q_n - p_{n+1} q_{n+1}) = 0, \tag{21}$$

since we have periodic boundary conditions. No matter what the initial amount of money in the system is, the system will go to the equilibrium where precisely that amount is needed—the final I's are fixed by the initial money supply. The total amount of money in the economy is irrelevant, since the utility and amount of goods exchanged in the final equilibrium does not depend on that. However, as previously described, changes in the amount of money have interesting transient effects.

V. CONCLUSION

Here we considered a simple toy model with simple monopolistic agents. In general, economy deals with complicated heterogeneous networks of agents, with complicated links to one another, representing the particular ''games'' they play with one another. We submit that the general picture remains the same. At each trade, the agents evaluate the value of money, by analyzing their particular local situation, and act accordingly. The prices charged by the agents will be constrained by those of the interacting agents. It would be interesting to study the formation and stability of markets where very many distributed players are interested in the same goods, but not generally interacting directly with one another. Indeed, we have considered an allied model with a market structure introduced in a related, more explicitly economics-oriented discussion [4].

Modifications of this network model may also provide a toy laboratory for the study of the effects of the introduction of the key financial features of credit and bankruptcy as well as the control problems posed by the governmental role in varying the money supply.

[1] G. Debreu, *Theory of Value* (Wiley, New York, 1959).

[2] N. Kiyotaki and R. Wright, J. Economic Theory **53**, 215 (1991).

[3] A. Trejos and R. Wright, J. Political Economy **103**, 118 (1995).

[4] A detailed economics-oriented discussion of the search theoretic and other approaches to the theory of money is given in our companion paper [P. Bak, S. F. Nørrelykke, and M. Shubik (in preparation)] where the concept of market is made operational.

[5] W. S. Jevons, *Money and the Mechanism of Exchange* (MacMillan, London, 1875).

[6] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).

[7] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[8] This is mathematically equivalent to differentiating Eq. (8) with respect to $I_{n,\tau}$ [with the expressions in Eqs. (9), (10), and (11) substituted for $p_{n,\tau}$, $q_{n+1,\tau}$, and $p_{n+1,\tau}$] and demanding it be zero.

[9] The solution to Eq. (17) for a delta-function initial condition is $h(x,t) = (C/\sqrt{4\pi D t}) \Sigma_{j=-\infty}^{\infty} \exp[-(x-jL-vt)^2/4Dt]$, where L is the system size and $x \in [0,L]$.